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Exercises for 'Functional Analysis 2' [MATH-404]

(28/04/2025)

Ex 9.1 (The chain rule on Banach spaces)
Let X, Y, Z be Banach spaces and $U \subset X$, $V \subset Y$ be open. Assume that $F : U \to Y$ is differentiable in $x_0 \in U$ with $F(x_0) \in V$ and that $G : V \to Z$ is differentiable in $F(x_0)$. Show that $G \circ F : U \to Z$ is differentiable in x_0 with

$$(G \circ F)'(x_0) = G'(F(x_0))F'(x_0).$$

Hint: Using little-o notation simplifies the calculations.

Ex 9.2 (Fréchet-differentiability)
a) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function. Show that

$$u \mapsto F(u) = \int_0^1 f(u(x))\,\mathrm{d}x$$

is differentiable on $C([0,1])$ equipped with the maximum norm and compute its derivative. Is the derivative continuous?

b) Let $k \in C([0,1] \times [0,1])$, $f \in C^1([0,1] \times \mathbb{R})$ and consider the **Hammerstein operator**

$$u \mapsto F(u) = \int_0^1 k(\,\cdot\,, x) f(x, u(x))\,\mathrm{d}x.$$

Check that $F \colon C([0,1]) \to C([0,1])$ is continuously differentiable.

Ex 9.3 (Gâteaux=Fréchet for Lipschitz functions on finite-dimensional spaces*)
Suppose $F \colon X \to Y$ is a Lipschitz function from a finite-dimensional Banach space X to a (possibly infinite-dimensional) Banach space Y. Prove that if F is Gâteaux-differentiable at some point x, then it is also Fréchet-differentiable at that point.

Ex 9.4 (Gâteaux-differentiability)
a) We saw in the lectures that the function $F : L^2([0,1]) \to L^2([0,1])$ defined by $F(u)(x) = \cos(u(x))$ is not differentiable in 0 (actually it is nowhere differentiable). Prove that it is Gâteaux-differentiable on $L^2([0,1])$ with $\delta F(u)v = -(\sin \circ u) \cdot v$.

b) Show that if the function $u_0 \in C([0,1])$ is such that $|u_0|$ attains its maximum on $[0,1]$ at a single point t_0, the norm $u \mapsto \|u\| = \sup_{x \in [0,1]} |u(x)|$ on the Banach space $C([0,1])$ is Gâteaux differentiable at u_0 and for any direction $v \in C([0,1])$

$$\delta \|u_0\| v = v(t_0) \cdot \operatorname{sign} u_0(t_0).$$

c) [**difficult exercise, optional**] Show that the norm is not Gâteaux differentiable at u_0 if u_0 does not satisfy the above assumption.
Hint: Show that for any maximizer t_0 of $|u(t)|$ the map found in b) has to coincide with the Gâteaux-derivative.